Uniform Selection in Global Games

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Abstract

This paper brings together results which are required in order to extend the global games approach to settings where the game structure is endogenous. More precisely, it shows that the selection argument of Carlsson and van Damme [2] holds uniformly over appropriately controlled families of games. Those results also give proper justification for the inversion of limits which is often implicit in applied work taking comparative statics on the selected risk-dominant equilibrium.

KEYWORDS: global games, equilibrium selection, uniform selection, comparative statics, endogenous games

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1 Introduction

The global games framework, first proposed in Carlsson and van Damme [2], has been widely applied as a selection device in 2×2 coordination games[1]. Their first result is that if players have an information structure resulting from independent additive observational noise, then, the set of rationalizable strategies shrinks to a unique equilibrium as the amplitude of the noise goes to 0. Their second result is that the selected equilibrium is in fact to play the risk-dominant strategy studied in Harsanyi and Selten [4].

While it is true that global games have been widely used in applied work, there have been only few attempts to use the information structure of Carlsson and van Damme [2] in models of greater complexity than one shot two actions coordination games. A notable exception is Frankel, Morris and Pauzner [3] which proves selection results for a class of supermodular games in which the actions and the state of the world belong to the real line. Adding layers of decision making on top of a 2×2 coordination problem is another tempting direction in which to extend the results of Carlsson and van Damme [2]. The main hurdle towards that goal is that in such models, the payoffs of the coordination game will be endogenously determined: to use selection results in this setting, we need them to hold uniformly over the class of possible payoffs. Up to now however, available selection results all take the payoff structure as given; in other words, they hold pointwise while we need uniform selection. A simple example makes this point clearer (see Appendix A for more detailed examples).

Consider the problem of a principal trying to get her two agents to cooperate. The game has three periods: at time $t = 1$, the principal can invest in some capital k at a positive increasing cost $c(k)$. At time $t = 2$, the two agents observe k perfectly and then play a global game $\Gamma(\theta, k, \sigma)$ with actions $\{cooperate, defect\}$, where θ is the noisily observed state of the world, σ the amplitude of the noise and capital k parameterizes players' payoffs. At time $t = 3$, the principal gets a payoff depending on whether the agents cooperated or not. What is the optimal capital stock k^*_σ the principal should purchase? How do k^*_σ and the principal's payoff vary as σ goes to zero?

To solve her optimization problem, the principal must form some belief about her agents' behavior. When the state of the world is common knowledge, because of multiplicity of

[1]See Morris and Shin [7] for a literature review.

equilibria, this problem is not well defined. This motivates the use of a global game information structure. The typical global games selection results state that for a given capital stock k, the agents will cooperate if and only if the state of the world is above some threshold $\theta_\sigma(k)$ and that as σ goes to zero, $\theta_\sigma(k)$ converges to the risk-dominant threshold $\theta^{RD}(k)$. Uniqueness of equilibrium makes the principal's problem well defined. Assume there is an optimal amount of capital k_0^* the principal would choose if there was no noise and the agents used the risk-dominant threshold $\theta^{RD}(k)$. Is it true that k_σ^* converges to k_0^* as σ goes to zero? Is it true that the principal's payoff is continuous in σ?

Under general circumstances, the answer to these questions is affirmative, however as the counter-example of Figure 1 shows, pointwise convergence of $\theta_\sigma(k)$ is not sufficient for these results to hold. In this counter-example the cooperation threshold $\theta_\sigma(k)$ converges pointwise to a constant threshold equal to $\frac{1}{2}$, but does not converge uniformly. In fact, there is always a capital stock such that the players' cooperation threshold is $\frac{1}{4}$. If agents behaved according to Figure 1, the principal might choose a capital stock $k_\sigma^* = 1 - \frac{\sigma}{2}$ for all $\sigma > 0$, but at the limit she would choose $k_0^* = 0$ since capital is costly. To show that in fact k_σ^* does converges to k_0^*, we need to prove uniform convergence of $\theta_\sigma(k)$ over the set of possible capital stocks as σ goes to 0.

The goal of this paper is to provide uniform selection results over general families of payoffs. Section 2 defines the classes of payoffs over which we will prove uniform selection. Section 3, which constitutes the core of the paper, proves the main selection results. Section 4 concludes. Appendix A provides explicit examples for which uniform selection either fails, or isn't obvious. Appendix B extends the analysis to symmetric games with a continuum of players. Proofs are contained in Appendix C unless mentioned otherwise.

2 Choosing an appropriate payoff class

This section introduces the class of games that we will be studying. Keeping with the framework of Carlsson and van Damme [2], the paper focuses on two actions two players games. Results presented in coming sections extend to symmetric games with two actions and a continuum of players, such as those reviewed by Morris and Shin [7]. The extension is presented in Appendix B.

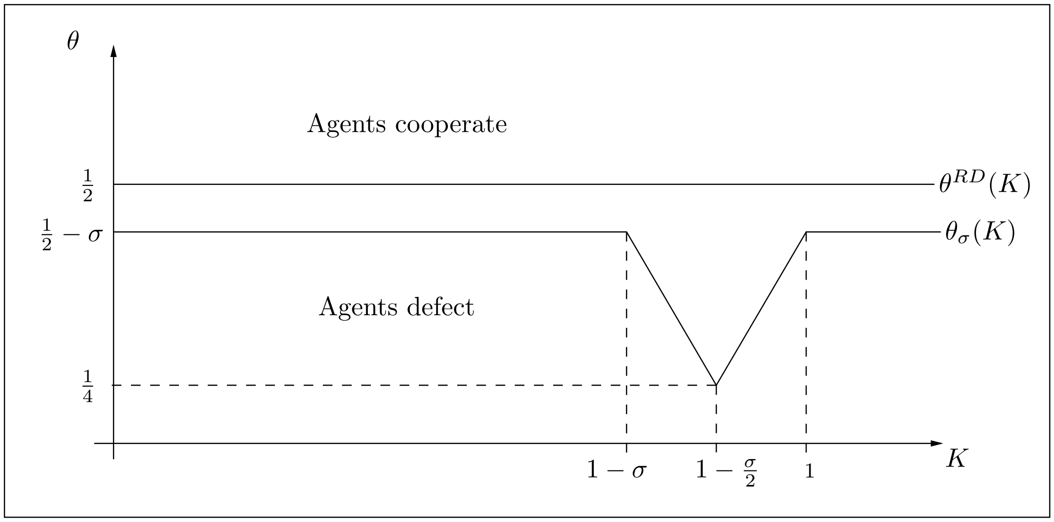

Figure 1: A sequence of thresholds converging pointwise but not uniformly. Agents cooperate above the threshold and defect below.

We consider 2×2 games, with players $i \in \{1, 2\}$, actions $a \in \{C, D\}$ and payoffs that depend continuously on a state of nature $\theta \in I$, where I is an interval of $\mathbb{R}$. Payoffs are denoted by

	C	**D**
C	$w^i_{11}(\theta)$	$w^i_{12}(\theta)$
D	$w^i_{21}(\theta)$	$w^i_{22}(\theta)$

where i is the row player. Both players get signals $x_i = \theta + \sigma\varepsilon_i$, where ε_1 and ε_2 are independent random variables with support $[-1, 1]$, and θ is a random variable with a C^1 distribution f_θ and convex support.

Let $G(\theta)$ denote the game with *full information* at state θ and let Γ_σ be the global game with *noisy information*. Denote by w the payoff structure $(w^i_{11}, w^i_{12}, w^i_{21}, w^i_{22})_{i\in\{1,2\}}$. Pure strategies are functions $s : \mathbb{R} \mapsto \{C, D\}$. For completeness, mixed strategies will be considered by allowing players to privately observe independent random variables $\tilde{u}$ uniformly distributed on $[0, 1]$: mixed strategies can be viewed as functions $s : \mathbb{R} \times [0, 1] \mapsto \{C, D\}$. To eliminate multiple representations, the constraint is imposed that for all $x \in \mathbb{R}$, and $(u, u') \in [0, 1]^2$, whenever $u < u'$, then $\{s(x, u) = C \Rightarrow s(x, u') = C\}$. Pure strategies are also mixed strategies, which do not depend on the random variable $\tilde{u}$.

In order to apply global games techniques to endogenous payoff structures we need to prove selection results holding uniformly over some family $\mathbf{\Lambda}$ of possible payoff functions. Rather than dealing with the problem on a case by case basis, the goal of this paper is to prove uniform selection results holding for general classes of payoffs. However, choosing an appropriate reference class of payoff functions $\mathbf{\Lambda}$ is delicate. While well behaved classes allow for simple proofs and fewer cases, they also limit the applicability of the results. This section defines and motivates the fairly general reference class of payoff functions we will use. We must first introduce a few assumptions and definitions.

Assumption 1 *For any state of the world θ, the game $G(\theta)$ has pure strategy equilibria. The set of equilibria is either $\{(C,C)\}$,$\{(D,D)\}$ or $\{(C,C),(D,D)\}$.*

This assumption rules out games of "matching pennies" and ensures that there exists a fixed order on actions such that for all states of the world, the game $G(\theta)$ is either dominance solvable or supermodular with respect to the aforementioned order.

Assumption 2 (increasing differences in the state of the world) *The game has increasing differences in θ:*
$\forall i \in \{1,2\}$, both $a^i(\theta) \equiv w^i_{12}(\theta) - w^i_{22}(\theta)$ and $b^i(\theta) \equiv w^i_{11}(\theta) - w^i_{21}(\theta)$ are strictly increasing in θ.

Assumption 3 (Dominance regions) *Let w be a payoff structure satisfying Assumptions 1 and 2. There exist thresholds $\underline{\theta}_i$ and $\overline{\theta}_i$ solutions to*

$$w^i_{12}(\overline{\theta}_i) - w^i_{22}(\overline{\theta}_i) = 0 \quad \textit{and} \quad w^i_{11}(\underline{\theta}_i) - w^i_{21}(\underline{\theta}_i) = 0.$$

Together, Assumptions 1, 2, and 3 insure that whenever $G(\theta)$ has multiple equilibria, either $(-\infty, \theta]$ is included in the risk-dominance region of (D,D) or $[\theta, \infty)$ is included in the risk-dominance region of (C,C). This is the unidimensional equivalent of Carlsson and van Damme's assumption that states should be connected to dominance regions by a path included in the risk-dominant region of one equilibrium.

Definition 1 (differences in actions) *Given a payoff structure w, we define $h^i_w(\theta) = b^i_w(\theta) - a^i_w(\theta) = w^i_{11} - w^i_{12} - w^i_{21} + w^i_{22}$.*

Whenever $h_w^i(\theta) > 0$, then the game has strictly increasing differences in actions for player i at θ, that is, $w_{11}^i - w_{21}^i > w_{12}^i - w_{22}^i$. Supermodularity requires that for all $\theta \in I$, $h_w^i(\theta) \geq 0$ should hold. While Assumption 1 does imply increasing differences in actions over some intermediate range of states, assuming full supermodularity is in fact quite restrictive. Consider for instance the symmetric game :

	C	**D**
C	θ	$\gamma\theta$
D	M	$M/2$

where payoffs are given for the row player, $\gamma < 1/2$, and $\theta \in \mathbb{R}$. This game satisfies Assumption 1 but not supermodularity.

Definition 2 (modulus of continuity) *A function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus of continuity if and only if it is continuous, strictly increasing and $\rho(0) = 0$. A funtion $g : \mathbb{R} \to \mathbb{R}$ has a modulus of continuity ρ if and only if*

$$\forall (x, y) \in \mathbb{R}^2, \quad |g(x) - g(y)| \leq \rho(|x - y|).$$

We will require payoff functions to share a common modulus of continuity. We know from the Arzelà-Ascoli theorem[2] that this is in fact a compactness assumption. It is less restrictive than assuming that payoff functions are Lipschitz continuous with a common rate R. Because utility functions commonly used in economics typically satisfy the Inada conditions, they are not Lipschitz continuous; however they do admit a modulus of continuity.

Definition 3 (rates) *Let g be some function from I to $\mathbb{R}$. We define the upper and lower rates of g at θ by,*

$$\frac{\partial^+ g}{\partial \theta}(\theta) = \limsup_{\theta' \to \theta} \frac{g(\theta') - g(\theta)}{\theta' - \theta} \qquad \text{and} \qquad \frac{\partial^- g}{\partial \theta}(\theta) = \liminf_{\theta' \to \theta} \frac{g(\theta') - g(\theta)}{\theta' - \theta}.$$

Those rates are always well defined although they might take infinite values. The lower and upper rates of g coincide at θ if and only if g is differentiable at θ.

[2] For a reference, see James Munkres, *Topology*, Prentice Hall, 2000.

Definition 4 (Reference payoff class) *Consider $\kappa > 0$, $\nu > 0$, $d > 0$, a modulus of continuity ρ, and a compact set $K \subset \mathbb{R}$. We denote by $\Lambda_{\kappa,\nu,d,\rho,K}$ the class of payoff structures w such that,*

1. *w satisfies Assumptions 1, 2 and 3*

2. *$\forall i \in \{1,2\}$, a_w^i and b_w^i have lower rates greater than $\kappa > 0$ over $[\underline{\theta}_i(w) - \nu, \overline{\theta}_i(w) + \nu]$*

3. *$\forall i \in \{1,2\}$, $[\underline{\theta}_i(w), \overline{\theta}_i(w)] \subset K$*

4. *payoff functions corresponding to w have a modulus of continuity ρ*

5. *$\forall i \in \{1,2\}$, $\overline{\theta}_i(w) - \underline{\theta}_i(w) > d$.*

As noted previously, the payoff structures we are considering are not necessarily supermodular. However, as the following lemma shows, if $w \in \Lambda_{\kappa,\nu,d,\rho,K}$ it does satisfy increasing differences in actions over the range of states of the world for which there are potentially multiple equilibrium action.

Lemma 1 (increasing differences in actions) *Consider a class $\Lambda_{\kappa,\nu,d,\rho,K}$ of payoff structures and define $\hbar \equiv d\kappa/2$ and $r = \rho^{-1}(d\kappa/8)$. Then, for all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$,*

$$\forall i \in \{1,2\}, \ \ \forall \theta \in [\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r], \quad h_w^i(\theta) \geq \hbar > 0.$$

3 Uniform selection

In this section we prove the two main results of the paper:

Joint selection: There exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, and all payoff structures w in $\Lambda_{\kappa,\nu,d,\rho,K}$, the game $\Gamma_\sigma(w)$ has a unique rationalizable pair of strategies.

Uniform convergence: The selected equilibrium converges uniformly over $\Lambda_{\kappa,\nu,d,\rho,K}$ to the risk-dominant equilibrium associated with each payoff profile.

The logic of the proof is the following: we first prove that for a noise term σ small enough, Assumption 1 implies that the game Γ_σ exhibits monotone best response and has extreme monotone Nash equilibria; then we prove joint and uniform selection results by showing that these extreme equilibria are characterized by two real equations whose solutions are well behaved in the underlying payoff structure.

3.1 Structural results

In this section we consider a payoff class $\Lambda_{\kappa,\nu,d,\rho,K}$ and show that there exists $\overline{\sigma} > 0$, such that for all $\sigma \in (0, \overline{\sigma})$, and all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$, the set of rationalizable strategies of $\Gamma_\sigma(w)$ is bounded by extreme monotone Nash equilibria. To do this, we first define a natural order on strategies denoted by $\preccurlyeq$. We then show that for σ small enough, the best response mapping is increasing with respect to $\preccurlyeq$, and preserves the monotonicity of strategies.

Note that since we do not assume supermodularity, the results of van Zandt and Vives [8] for Bayesian games do not apply. Also, although there is a consistent order over $\{C, D\}$ such that for all $\theta \in I$, $G(\theta)$ satisfies the single-crossing property of Milgrom and Shannon [6], this does not imply that the Bayesian game Γ_σ satisfies the single-crossing property. Finally, because we do not make a monotone likelihood-ratio assumption on the signalling structure, the results of Athey [1] cannot be applied, which is why we must limit the amplitude σ of the noise term. In fact the essence of Proposition 2 is to show that the monotone likelihood-ratio property holds approximately as σ goes to 0.

Definition 5 (monotone strategies) *A strategy s is said to be monotone if it is a pure strategy, and admits a threshold x_s such that,*

$$x < x_s \Rightarrow s(x) = D \quad \text{and} \quad x > x_s \Rightarrow s(x) = C.$$

A monotone strategy of threshold x will be denoted s_x and inversely, the threshold of a monotone strategy s will be denoted x_s.

Definition 6 (ordered strategies) *Let $\preccurlyeq$ denote the partial order on pure an mixed strategies defined by*

$$s \preccurlyeq s' \iff \forall (x, u) \in \mathbb{R} \times [0, 1], \quad s(x, u) = C \Rightarrow s'(x, u) = C.$$

Given a pair of strategies (s, s') such that $s \preccurlyeq s'$, we denote by $[s, s']$ the set of all strategies s'' such that $s \preccurlyeq s'' \preccurlyeq s'$.

Let BR^i denote the best-response correspondence[3] of player i. Given her opponent's strategy s and her signal x_i, player i's expected payoffs upon cooperation an defection are

$$\Pi_C(x_i, s) = \mathbf{E}\left[w^i_{12} + \{w^i_{11} - w^i_{12}\}\mathbf{1}_{s=C}|x_i, s\right] \tag{1}$$

$$\Pi_D(x_i, s) = \mathbf{E}\left[w^i_{22} + \{w^i_{21} - w^i_{22}\}\mathbf{1}_{s=C}|x_i, s\right]. \tag{2}$$

Define $\Delta^w_i(x_i, s)$ by $\Delta^w_i(x_i, s) \equiv \Pi_C(x_i, s) - \Pi_D(x_i, s)$.

Lemma 2 (highest and lowest best-response) *For any strategy s, $BR^i(s)$ admits a highest and lowest element with respect to $\preccurlyeq$, respectively denoted $BR^{i,H}(s)$ and $BR^{i,L}(s)$.*

Proof: A best response to s must prescribe C whenever $\Delta^w_i(x_i, s) > 0$ and D whenever $\Delta^w_i(x_i, s) < 0$. The best-response will have multiple elements if and only $\Delta^w_i(x_i, s) = 0$ over a set with non-zero mass. The existence of the highest and lowest best reply is proven by construction: $BR^{i,H}(s)$ prescribes C if and only if $\Delta^w_i(x_i, s) \geq 0$, while $BR^{i,L}(s)$ prescribes C if and only if $\Delta^w_i(x_i, s) > 0$. ■

We now show that for σ small enough, best-reply correspondences are well behaved with respect to the order $\preccurlyeq$ and maintain the monotonicity of strategies.

Proposition 1 (monotone best response) *For all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$ and $\sigma \in [0, r/2]$, the game $\Gamma_\sigma(w)$ has monotone best response. That is, for any s and s',*

$$s' \preccurlyeq s \quad \Rightarrow \quad \left\{BR^{i,H}(s') \preccurlyeq BR^{i,H}(s) \quad \textit{and} \quad BR^{i,L}(s') \preccurlyeq BR^{i,L}(s)\right\}.$$

Proposition 2 (monotone strategies) *There exists $\overline{\sigma} > 0$ such that for all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$, $\sigma \in (0, \overline{\sigma})$, and any monotone strategy s_x, then there exists $x' \in \mathbb{R}$ such that $BR^i(s_x) = \{s_{x'}\}$. Moreover, the threshold x' associated with $BR^i(s_x)$ is a continuous function of x.*

[3]For readability, the notation omits the dependency of BR^i on both w and σ.

We can now state our first theorem which shows the existence of extreme monotone strategies and greatly simplifies the rest of the analysis.

Theorem 1 (extreme strategies) *There exists $\overline{\sigma} > 0$ such that for all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$ and all $\sigma \in (0, \overline{\sigma})$, the set of rationalizable strategies of game $\Gamma_\sigma(w)$ is bounded by extreme monotone equilibria.*

Proof: Pick $\overline{\sigma}$ such that Propositions 1 and 2 hold. Let R_i denote the set of rationalizable strategies of player i. R_i is the biggest fixed set of $BR^i \circ BR^{-i}$. We know from Proposition 1 that $BR^i \circ BR^{-i}$ is monotonically increasing with respect to the partial order $\preccurlyeq$, and we know from Proposition 2 that it preserves the monotonicity of strategies. Thus we can entirely replicate the construction given by Milgrom and Roberts [5] and Vives [9] for supermodular games. ■

Using this result, the rest of the analysis can now focus on extreme threshold-form strategies. Note that monotone strategies are pure strategies.

3.2 Selection

In this section we prove *joint selection* and *uniform convergence*. Consider a class of payoffs $\Lambda_{\kappa,\nu,d,\rho,K}$, we want to characterize the set of rationalizable strategies of global games $\Gamma_\sigma(w)$ with $w \in \Lambda_{\kappa,\nu,d,\rho,K}$ and σ small. The first step is to use Theorem 1, which implies that there exists $\overline{\sigma} > 0$ such that whenever $\sigma \in (0, \overline{\sigma})$, we only need to study monotone Nash equilibria to prove that there is a unique rationalizable equilibrium. A monotone equilibrium is characterized by a pair of thresholds (x_i, x_{-i}) such that,

$$\Delta_i^w(x_i, x_{-i}, \sigma) = 0, \quad \text{for } \ i \in \{1, 2\}. \tag{3}$$

This equation derives from the fact that in the presence of observational noise, players' payoffs must be continuous in their signal. This implies that at a threshold point, players must be indifferent between their two actions, which provides extra restrictions that must be satisfied in equilibrium. Hence, to study equilibrium selection, it is equivalent to study the behavior of the set of indifference equations (3). We have

(4)
$$\Delta_i^w(x_i,x_{-i},\sigma)=\int_{-\infty}^{+\infty}\left[a^i(\theta)F_\varepsilon\left(\frac{x_{-i}-\theta}{\sigma}\right)+b^i(\theta)G_\varepsilon\left(\frac{x_{-i}-\theta}{\sigma}\right)\right]\frac{f_\varepsilon\left(\frac{x_i-\theta}{\sigma}\right)f_\theta(\theta)}{\int_{-\infty}^{+\infty}f_\varepsilon\left(\frac{x_i-\theta}{\sigma}\right)f_\theta(\theta)\mathrm{d}\theta}\mathrm{d}\theta.$$

Let us do the change in variable $u=\frac{x_i-\theta}{\sigma}$. Noting that f_ε only puts mass on the $[-1,1]$ interval, we obtain

$$\begin{aligned}\Delta_i^w(x_i,x_{-i},\sigma) &= \int_{-1}^{+1}\Bigg[a^i(x_i-\sigma u)F_\varepsilon\left(u+\frac{x_{-i}-x_i}{\sigma}\right)\\ &+ b^i(x_i-\sigma u)G_\varepsilon\left(u+\frac{x_{-i}-x_i}{\sigma}\right)\Bigg]\frac{f_\varepsilon(u)f_\theta(x_i-\sigma u)}{\int_{-1}^{1}f_\varepsilon(u)f_\theta(x_i-\sigma u)\mathrm{d}u}\mathrm{d}u.\end{aligned}$$

The above expression has a $(x_i-x_{-i})/\sigma$ term which blows up as σ goes to 0. In order to have functions that have a continuous limit as σ goes to 0, we define α by $x_{-i}=x_i+\alpha\sigma$ and abuse notations slightly by denoting $\Delta_i^w(x_i,\alpha,\sigma)\equiv\Delta_i^w(x_i,x_{-i},\sigma)$. We obtain

(5)
$$\begin{aligned}\Delta_i^w(x_i,\alpha,\sigma) &= \int_{-1}^{+1}\Big[a^i(x_i-\sigma u)F_\varepsilon(u+\alpha)\\ &+ b^i(x_i-\sigma u)G_\varepsilon(u+\alpha)\Big]\frac{f_\varepsilon(u)f_\theta(x_i-\sigma u)}{\int_{-1}^{1}f_\varepsilon(u)f_\theta(x_i-\sigma u)\mathrm{d}u}\mathrm{d}u.\end{aligned}$$

We can now think of a monotone equilibrium as a pair (x_i,α), such that $\Delta_i^w(x_i,\alpha,\sigma)=\Delta_{-i}^w(x_i+\alpha\sigma,-\alpha,\sigma)=0$. The essence of our proof technique is to show that this equation has a unique solution and that it is well behaved in w. To do this we need to understand how Δ_i^w varies with x_i, α and σ.

More precisely Lemmas 3, 4 and 5 establish that over a specific range for parameters (x_i,α) which includes any threshold equilibrium for σ small enough: $\Delta_i^w(x_i,\alpha,\sigma)$ is strictly increasing in x_i and strictly decreasing in α with rates bounded away from 0 independently of either σ or w; as σ goes to 0; $\Delta_i^w(x,\alpha,\sigma)$ converges uniformly over $(x,\alpha)\in\mathbb{R}^2$ at a rate that depends only on $\Lambda_{\kappa,\nu,d,\rho,K}$. As in Lemma 1, given a class $\Lambda_{\kappa,\nu,d,\rho,K}$, we define $\hbar\equiv d\kappa/2$ and $r=\rho^{-1}(d\kappa/8)$.

Lemma 3 *Given a class of payoff structures $\Lambda_{\kappa,\nu,d,\rho,K}$, then there exists $\overline{\sigma}>0$ such that,* *(i) For all payoff structures $w \in \Lambda_{\kappa,\nu,d,\rho,K}$ and $\sigma \in (0, \overline{\sigma})$, all monotone equilibria (x_i, α) of game $\Gamma_\sigma(w)$ are such that $\alpha \in [-2, 2]$.*

(ii) For all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$, $\sigma \in (0, \overline{\sigma})$ and $\alpha \in [-2, 2]$,

$$\forall\, x_i \in [\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r], \quad \frac{\partial \Delta_i^w}{\partial x_i}(x_i, \alpha, \sigma) \geq \kappa/2 > 0.$$

(iii) For all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$, $\sigma \in (0, \overline{\sigma})$, $x_i \in [\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r]$ and $\alpha \in [-2, 2]$

$$\frac{\partial \Delta_i^w(x_i, \alpha, \sigma)}{\partial \alpha} \leq -\hbar \int_{-1}^{1} f_\varepsilon(u + \alpha) \frac{f_\varepsilon(u) f_\theta(x_i - \sigma u)}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x_i - \sigma u)\, du}\, du \leq 0.$$

Lemma 3 guarantees that the solutions to $\Delta_i^w(x_i, \alpha, \sigma) = 0$ will be well-behaved when w and σ vary. This allows us to state our first selection result which says that for all σ less than some $\overline{\sigma}$ small enough, selection happens jointly for all games with payoffs in $\Lambda_{\kappa,\nu,d,\rho,K}$. It does not discuss how the selected equilibria behave as σ goes to 0. This will be the object of Theorem 3.

Theorem 2 (joint selection) *There exists $\overline{\sigma} > 0$ sufficiently small such that for all $\sigma \in (0, \overline{\sigma})$ and $w \in \Lambda_{\kappa,\nu,d,\rho,K}$, all global games $\Gamma_\sigma(w)$ have a unique pair of rationalizable strategies.*

Proof: Take $\overline{\sigma}$ such that Lemma 3 holds. We know from Theorem 1 that the set of rationalizable strategies is bounded by monotone equilibria, so it suffices to show there is a unique monotone equilibrium. Such an equilibrium is characterized by a pair (x_i, α) such that, $\Delta_i^w(x_i, \alpha, \sigma) = \Delta_{-i}^w(x_i + \alpha\sigma, -\alpha, \sigma) = 0$. From parts *(ii)* and *(iii)* of Lemma 3, we know that Δ_i^w is strictly increasing in x_i and weakly decreasing in α. Thus, the first equilibrium condition $\Delta_i^w(x_i, \alpha) = 0$ implicitly defines a function $\alpha(x_i)$ that is weakly increasing in x_i. Replace that in the other equilibrium condition: x_i is such that $\Delta_{-i}^w(x_i + \alpha(x_i)\sigma, -\alpha(x_i), \sigma) = 0$. Define $\zeta(x_i, \sigma, w) \equiv \Delta_{-i}^w(x_i + \alpha(x_i)\sigma, -\alpha(x_i), \sigma)$. Lemma 3 implies that this function is strictly increasing in x_i which implies there is at most a unique value x_i satisfying $\zeta(x_i, \sigma, w) = 0$. Existence results from Assumption 3. Using Theorem 1 we conclude that there is a unique pair of rationalizable strategies. ■

We now proceed to show that as σ goes to 0, this uniquely selected equilibrium converges to the risk-dominant strategy uniformly over $\Lambda_{\kappa,\nu,d,\rho,K}$. Since we already know pointwise convergence, this is equivalent to showing that the zeroes of $\Delta_i^w(x_i, \alpha, \sigma)$ converge uniformly as σ goes to 0. For this, we need to show that $\Delta_i^w(\cdot, \cdot, \sigma)$ converges uniformly, and that $\frac{\partial \Delta_i^w}{\partial \alpha}$ is uniformly bounded away from 0.

Lemma 4 *Consider a class $\Lambda_{\kappa,\nu,d,\rho,K}$, there exists $N > 0$ such that for all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$,*

$$|\Delta_i^w(x, \alpha, \sigma) - \Delta_i^w(x, \alpha, 0)| \leq N \max\{\rho(\sigma), \sigma\}.$$

Without loss of generality, we can always assume that $\rho(\sigma) \geq \sigma$. Indeed if a function has a modulus of continuity ρ, it has a modulus of continuity $\tilde{\rho}$ for all $\tilde{\rho}$ greater than ρ.

Lemma 5 *Given a family of payoff structures $\Lambda_{\kappa,\nu,d,\rho,K}$, there exists $\overline{\sigma} > 0$, $\lambda > 0$ and $\eta > 0$ such that whenever $\sigma \in (0, \overline{\sigma})$ and $w \in \Lambda_{\kappa,\nu,d,\rho,K}$, any monotone equilibrium (x_i, α) of $\Gamma_\sigma(w)$ is such that,*

1. $\alpha \in [-2 + \lambda, 2 - \lambda]$
2. $\forall x_i \in [\underline{\theta}_i - \nu, \overline{\theta}_i + \nu]$, $\forall \alpha \in [-2 + \lambda, 2 - \lambda]$,

$$\frac{\partial \Delta_i^w(x_i, \alpha, \sigma)}{\partial \alpha} < -\eta$$

3. *Denote by $\alpha(x_i, w)$ the implicit function solving $\Delta_i^w(x_i, \alpha, \sigma) = 0$. For all x_i in $[\underline{\theta}_i - \nu, \overline{\theta}_i + \nu]$, $\alpha(x_i, w)$ is $(\frac{4}{\eta})$-Lipschitz in w, with respect to the norm on payoff structures defined by, $||w - \tilde{w}|| \equiv \max_{i,j,k \in \{1,2\}^3} ||w_{jk}^i - \tilde{w}_{jk}^i||_\infty$.*

We can now state the main result of the paper.

Theorem 3 (uniform convergence) *Consider a class of payoffs $\Lambda_{\kappa,\nu,d,\rho,K}$. We know from Theorem 2 that for $\sigma \in (0, \overline{\sigma})$, all games $\Gamma_\sigma(w)$ have a unique pair of rationalizable strategies, with thresholds $(x_i(w, \sigma), x_{-i}(w, \sigma))$. As σ goes to 0, the equilibrium threshold $x_i(w, \sigma)$ converges uniformly over $\Lambda_{\kappa,\nu,d,\rho,K}$ to the risk-dominant threshold. More formally,*

$$\lim_{\sigma \to 0} \max_{w \in \Lambda_{\kappa,\nu,d,\rho,K}} |x_i(w, \sigma) - x_i(w, 0)| = 0.$$

Proof: First, from Lemmas 4 and 5, we know that $|\Delta_i^w(x,\alpha,\sigma) - \Delta_i^w(x,\alpha,0)| < N\rho(\sigma)$ and that $\frac{\partial \Delta}{\partial \alpha} < -\eta$. This implies that the solution $\alpha(x,\sigma)$ to $\Delta_i^w(x,\alpha,\sigma) = 0$ must satisfy

$$|\alpha(x,w,\sigma) - \alpha(x,w,0)| \leq \frac{N\rho(\sigma)}{\eta}. \tag{6}$$

Recall the definition, $\zeta(x,w,\sigma) \equiv \Delta_{-i}^w(x + \alpha(x,w,\sigma)\sigma, -\alpha(x,w,\sigma), \sigma)$. A real number x is an equilibrium threshold of $\Gamma_\sigma(w)$ if and only if $\zeta(x,w,\sigma) = 0$. For all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$, the functions Δ_i^w share a common modulus of continuity. Hence inequality (6) implies that $\zeta(\cdot,\cdot,\sigma)$ converges uniformly to $\zeta(\cdot,\cdot,0)$. From Lemma 3 we know that $\frac{\partial^- \zeta}{\partial x} > \kappa/2$. This yields that

$$\max_{w \in \Lambda_{\kappa,\nu,d,\rho,K}} |x_i(w,\sigma) - x_i(w,0)| \leq \frac{2}{\kappa} \max_{w \in \Lambda_{\kappa,\nu,d,\rho,K}} ||\zeta(\cdot,w,\sigma) - \zeta(\cdot,w,0)||_\infty$$

which concludes the proof since $\zeta(\cdot,\cdot,\sigma)$ converges uniformly as σ goes to 0. ■

Theorem 3 implies that in the example given in the introduction, the principal's optimal level of capital stock for σ positive does converge to the optimal capital stock in the risk-dominant equilibrium, as long as the family of payoffs indexed by k belongs to some regular class $\Lambda_{\kappa,\nu,d,\rho,K}$. See Appendix A.1 for an example where this condition is not satisfied and uniform selection fails.

The next proposition deals with the continuity of the selected equilibrium with respect to the payoff structure. This continuity result is useful in applications, for instance to ensure the existence of maxima at σ small but strictly positive.

Proposition 3 (continuous selection) *Consider a class of payoff structures $\Lambda_{\kappa,\nu,d,\rho,K}$. There exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0,\overline{\sigma})$, the selected equilibrium threshold $x_i(w,\sigma)$ is a continuous function of the payoff structure w.*

4 Conclusion

The approach of global games taken in this paper stresses the fact that when players base their actions on noisy continuous signals, their payoffs should be continuous in those signals. This gives additional constraints on equilibria which can be exploited to prove uniform selection results over fairly general classes of payoffs.

These uniform selection results are handy tools for applied economists wishing to use the global games approach to study more intricate game structures. For instance, in the agency problem presented in the introduction, we have sufficient conditions guaranteeing that the limit of the principal's behavior is indeed the best response to the limit of the agents' behavior.

Finally, because these uniform selection results hold over general classes of payoffs, they also allow to apply global games selection recursively and may prove useful to extend the use of such an information structure to dynamic games.

Appendix A: Two examples

This section presents two examples respectively illustrating that: mistakenly assuming uniform selection may generate the wrong predictions; and that simpler proofs of uniform selection based on the monotonicity of thresholds are often not available.

A.1 A failure of uniform selection

Consider the problem of a principal trying to get his two agents to tryout a new technology (action C), which is potentially more productive than the old one (action D), but is also more sensitive to exogenous macroeconomic variations (parameter θ, which we take normally distributed). When agents play C, the final output depends on macro conditions via a countably infinite number of channels. The principal can affect the game agents play by shutting off such channels of dependence. When the principal shuts off n channels, the game played by agents has symmetric payoffs w_n

	C	**D**
C	$g(n)\theta + 1$	$g(n)\theta - 2$
D	0	0

where $n \in \mathbb{N}$ and g is a decreasing, strictly positive function such that $g(0) = 1$. For any n and a noise level σ, this defines a game $\Gamma_\sigma(w_n)$. Let us denote $c(n)$ the strictly increasing cost of shutting down n channels. Assume that σ is small but positive, what is the optimal number of channels for the principal to shut down?

We distinguish two cases, whether $\lim_{n\to\infty} g(n) > 0$ (case 1) or $\lim_{n\to\infty} g(n) = 0$ (case 2). In case 1, the family of payoffs $\{w_n\}_{n\in\mathbb{N}}$ satisfies Definition 4, while in case 2 it doesn't (it fails requirement 2 and 3).

Under case 1, Theorem 3 holds. Hence, when σ is small, for any $n \in \mathbb{N}$, the player's behavior is close to the risk-dominant strategy, which is to play C if and only if

$$\theta \geq \frac{1}{2g(n)}.$$

This means that it is actually harder for players to coordinate on C when n increases. Since $c(n)$ is strictly increasing, for σ small, it will be optimal for the principal *not* to hedge the players from macroeconomic shocks.

Under case 2, Theorem 3 does not hold any more. In fact the following lemma holds

Lemma 6 *For any fixed value of $\sigma > 0$, there exist values of n large enough such that the game $\Gamma_\sigma(w_n)$ has* multiple *extreme equilibria, in one of which players coordinate on C with probability close to one.*

If $\lim_{n\to\infty} c(n)$ isn't too large this means it might actually be optimal for the principal to provide enough insurance so that it becomes possible for players to coordinate on C with high probability. Clearly, mistakenly assuming that uniform selection held would have led to the wrong prediction.

Proof of Lemma 6: This is simply an instance of lower hemicontinuity with respect to payoffs. The proof is still given for completeness. Since the game is symmetric, we look for symmetric equilibria characterized by a coordination threshold $x \in \mathbb{R}$. For $\sigma > 0$, equilibrium thresholds are characterized by the equation $\zeta(x, n, \sigma) = 0$, where,

$$\begin{aligned}\zeta(x,n,\sigma) &= \int_{-1}^{1} \{[g(n)(x-\sigma u)+1]\, G_\varepsilon(u) + [g(n)(x-\sigma u)-2]\, F_\varepsilon(u)\} \\ &\quad \times \frac{f_\theta(x-\sigma u) f_\varepsilon(u)}{\int_{-1}^{1} f_\theta(x-\sigma\tilde{u}) f_\varepsilon(\tilde{u})\mathrm{d}\tilde{u}} \mathrm{d}u\end{aligned}$$

Define $\Psi(x,u,\sigma) \equiv \frac{f_\theta(x-\sigma u) f_\varepsilon(u)}{\int_{-1}^{1} f_\theta(x-\sigma\tilde{u}) f_\varepsilon(\tilde{u})\mathrm{d}\tilde{u}}$ and note that Ψ is a probability distribution over

$[-1,1]$. We have

$$\zeta(x,n,\sigma) \;=\; \underbrace{g(n)\left(x+\sigma\int_{-1}^{1}u\Psi(x,u,\sigma)\mathrm{d}u\right)}_{\equiv\zeta_1(x,n,\sigma)}+\underbrace{\int_{-1}^{1}[3G_\varepsilon(u)-1]\Psi(x,u,\sigma)\mathrm{d}u}_{\zeta_2(x,\sigma)}$$

Given that θ is normally distributed, simple algebra shows that as x becomes large and positive, then $\Psi(x,u,\sigma)$ puts all of the probability mass on $u=1$. Similarly, when x is large and negative, then $\Psi(x,u,\sigma)$ puts all of the probability mass on $u=-1$. This implies that

$$\lim_{x\to-\infty}\zeta_2(x,\sigma)=1 \qquad \text{and} \qquad \lim_{x\to+\infty}\zeta_2(x,\sigma)=-2.$$

Hence pick any $\overline{x}>0$ large enough so that $\zeta_2(\overline{x},\sigma)<0$ and $\zeta_2(-\overline{x},\sigma)>0$. Since $\lim_{n\to\infty}g(n)=0$, there exists n^* large enough so that $\zeta(\overline{x},n,\sigma)<0$ and $\zeta(-\overline{x},n,\sigma)>0$. Since for any $n\in\mathbb{N}$,

$$\lim_{x\to-\infty}\zeta(x,n,\sigma)=-\infty \qquad \text{and} \qquad \lim_{x\to+\infty}\zeta(x,n,\sigma)=+\infty$$

we obtain that the game $\Gamma_\sigma(w_{n^*})$ has an equilibrium with threshold less than $-\overline{x}$ and an equilibrium with threshold greater than $\overline{x}$. Since $\overline{x}$ can be taken as large as wanted, this concludes the proof. ■

A.2 Non-monotonic comparative statics

Appendix A.1 makes the point that uniform selection isn't granted. One might however wonder whether in specific settings for which payoffs are indexed by a real variable k, simpler proofs of uniform selection based on the monotonicity in k of coordination thresholds $x_i(w_k,\sigma)$ are available. This section shows that such monotonicity shouldn't be typically expected by giving a natural example in which the risk-dominant threshold itself isn't monotonic.

Consider two firms deciding to switch technology or not in a setting with complementarities. Payoffs to the new technology depend on a parameter θ (normally distributed) while payoffs to the old technology depend on a common-knowledge parameter $k\in[0,+\infty)$. More precisely, payoffs w_k take the form

	C	**D**
C	θ	$\theta - k - k^2$
D	$-k$	$-k$

When k is large, then it is valuable for the players to coordinate on C. Indeed, under complete information, the Pareto efficient equilibrium is to switch technology if and only if $\theta \geq -k$. The worse the baseline option, the more valuable it is for players to switch technology.

The risk-dominant equilibrium on the other hand is to switch if and only if $\theta \geq -k+k^2/2$. Hence, as k increases, the switching threshold first decreases while $k \in [0, 1]$ and then increases over the range $[1, +\infty)$. Indeed, when the old technology gets worse, the gains from switching get larger, however the cost of switching alone also get larger at an increasing rate. These two effects result in a non-monotonic switching threshold: switching is easiest when the old technology is bad but not too bad. When the old technology is very bad, the players' fear freezes them into immobilism.

This example illustrates the point that in cases of interest, where considering the risk-dominant thresholds might reverse comparative statics under complete information, the coordination thresholds for $\sigma > 0$ are likely to be non-monotonic in the underlying parameter.

Appendix B: Extension to games with a continuum of players

Here we briefly outline why results presented in Section 3 still hold in symmetric games with a continuum of agents.

We consider games with a continuum of agents indexed by $t \in [0, 1]$. Each player has an action set $\{C, D\}$. All decisions are taken simultaneously. Let us denote by q the proportion of players choosing to play C. Players have identical payoffs which depend on their own action $a \in \{C, D\}$, the aggregate outcome q, and a state of the world θ, with convex support $I \subset \mathbb{R}$, and a C^1 density f_θ. Let these payoffs be denoted by $U_C(q, \theta)$ and $U_D(q, \theta)$. Before taking action, player t gets a signal $x_t = \theta + \sigma\varepsilon_t$, where ε_t has support in $[0, 1]$ and all draws are independent. We denote $\Gamma_\sigma(u)$ this global game.

We define the class of game structures $\mathcal{H}_{\kappa,\rho}$ as follows,

Definition 7 *Given a modulus of continuity ρ and a number $\kappa > 0$, we denote by $\mathcal{H}_{\kappa,\rho}$ the class of payoff structures U such that,*

1. *For all $q \in [0,1]$, the functions $U_C(q,\cdot)$ and $U_D(q,\cdot)$ have a modulus of continuity rho with respect to θ.*

2. *The mapping $m : (q,\theta) \mapsto U_C(q,\theta) - U_D(q,\theta)$ is strictly increasing in both θ and q with lower rates greater than κ.*

3. *There exist $\underline{\theta}$ and $\overline{\theta}$ in I such that $m(0,\overline{\theta}) > 0$ and $m(1,\underline{\theta}) < 0$.*

To simplify analysis, Definition 7 assumes that the games are fully supermodular. As in the case of two player games it is possible to work under the weaker assumption that the game has strictly increasing differences in the state of the world, and that at all states θ, the perfect information version of the game that players face is either dominance solvable or exhibits increasing differences in q. For the sake of concision, this appendix will not deal with that level of generality.

Because we have assumed that payoff structures were supermodular, it follows that game $\Gamma_\sigma(U)$ has extreme Nash equilibria which are symmetric. A proof identical to that of Proposition 2 shows that for σ small enough, these equilibria take a threshold form, meaning that there is a threshold x such that player t chooses C when $x_t > x$ and D when $x_t < x$.

Consider the incentives of player t when other players use threshold x. The proportion of people choosing C is $q = P\left[\varepsilon > \frac{x-\theta}{\sigma}\right]$. Thus payoffs are given by

$$\Pi_C(x_t, x, \sigma) = \int_I U_C\left(P\left[\varepsilon > \frac{x-\theta}{\sigma}\right], \theta\right) f(\theta|x_t)d\theta \tag{7}$$

$$\Pi_D(x_t, x, \sigma) = \int_I U_D\left(P\left[\varepsilon > \frac{x-\theta}{\sigma}\right], \theta\right) f(\theta|x_t)d\theta. \tag{8}$$

For x to be an equilibrium threshold, it must be that player is indifferent between C and D when $x_t = x$. Thus equilibrium is characterized by the equation

$$\Delta^U(x, x, \sigma) \equiv \int_I m\left(P\left[\varepsilon > \frac{x-\theta}{\sigma}\right], \theta\right) f(\theta|x_t)d\theta = 0. \tag{9}$$

Do the change in variable $u = \frac{x-\theta}{\sigma}$, equation (9) becomes,

$$\Delta^U(x, x, \sigma) \equiv \int_{-1}^{1} m\left(P\left[\varepsilon > u\right], x - \sigma u\right) \frac{f_\varepsilon(u) f_\theta(x - \sigma u)}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x - \sigma u)du} du = 0. \tag{10}$$

As in Section 3, we define Ψ_σ by

$$\Psi_\sigma(x,u) = \frac{f_\varepsilon(u) f_\theta(x-\sigma u)}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x-\sigma u) du}$$

Lemma 8 establishes that $\Psi_\sigma(u,x)$ converges uniformly to $f_\varepsilon(u)$ as σ goes to 0 and that $\frac{\partial \Psi}{\partial x}$ converges uniformly to 0.

We also have that $\frac{\partial m}{\partial x} > \kappa$. Finally because we consider only symmetric games, we know that the α term that we needed to consider in section 3 is equal to zero.

This implies that uniformly over any compact,

$$\lim_{\sigma \to 0} \Delta(x,x,\sigma) = \int_{-1}^{1} m\left(P\left[\varepsilon > u\right], x\right) f_\varepsilon(u) du = \int_{-1}^{1} m(u,x) du$$

and that the solutions of the equation $\Delta(x,x,\sigma) = 0$ converge to the risk-dominant equilibrium, that is, the solution of equation $\int_{-1}^{1} m(u,x)du = 0$.

More precisely Theorems 2 and 3 extend as follows,

Theorem 4 *Consider a class of payoffs $\mathcal{H}_{\kappa,\rho}$. There exists $\overline{\sigma}$ such that for all $\sigma < \overline{\sigma}$, and all $U \in \mathcal{H}_{\kappa,\rho}$, all games $\Gamma_\sigma(U)$ have a unique pair of rationalizable strategies, with threshold $x(U,\sigma)$. As σ goes to 0, the equilibrium threshold $x(U,\sigma)$ converges uniformly over $\mathcal{H}_{\kappa,\rho}$ to the risk-dominant equilibrium.*

Proof: The proofs are identical to those of Theorems 2 and 3. Joint selection is proven by showing that for some $\overline{\sigma}$, $\sigma \in (0,\overline{\sigma})$ implies that $\Delta(x,x,\sigma)$ is strictly increasing in x. Uniform convergence results from the fact that the rate of convergence of $x(U,\sigma)$ has an upper bound that depends only on k and ρ. ∎

Appendix C: Proofs

Proof of Lemma 1: For the first inclusion: we have by definition $b^i_w(\underline{\theta}_i) = 0$ and $a^i_w(\overline{\theta}_i) = 0$. Since b^i_w and a^i_w both have lower rates greater than κ, this implies that for all w,

$$\forall \theta \in [\underline{\theta}_i, \overline{\theta}_i], \quad h^i_w(\theta) = b^i_w(\theta) - a^i_w(\theta) = b^i_w(\theta) - b^i_w(\underline{\theta}_i) + a^i_w(\overline{\theta}_i) - a^i_w(\theta) \geq \kappa(\theta - \underline{\theta}_i) + \kappa(\overline{\theta}_i - \theta) \geq d\kappa. \quad (11)$$

Since all components of w have a modulus of continuity ρ, h_w^i has a modulus of continuity 4ρ. This implies that whenever $|x-y| \le \rho^{-1}(d\kappa/8)$, then $|h_w^i(x) - h_w^i(y)| < d\kappa/2$. For any $\theta \in [\underline{\theta}_i - \rho^{-1}(d\kappa/8), \overline{\theta}_i + \rho^{-1}(d\kappa/8)]$, there exists $\tilde{\theta}$ such that $\tilde{\theta} \in [\underline{\theta}_i, \overline{\theta}_i]$ and $|\theta - \tilde{\theta}| < d\kappa/2$. Using inequality (11) we get that,

$$\forall \theta \in [\underline{\theta}_i - \rho^{-1}(d\kappa/8)\ ,\ \overline{\theta}_i + \rho^{-1}(d\kappa/8)], \quad h_w^i(\theta) \ge h_w^i(\tilde{\theta}) - |h_w^i(\tilde{\theta}) - h_w^i(\theta)| \ge d\kappa/2$$

which concludes the proof. ■

Lemma 7 (rationalizable strategies) *For all $\sigma \ge 0$ and any strategy s, $BR^i(s) \subset [s_{\underline{\theta}_i - \sigma}, s_{\overline{\theta}_i + \sigma}]$. Moreover, a rationalizable strategy s has to belong to $\cap_{i \in \{1,2\}} [s_{\underline{\theta}_i - 2\sigma}, s_{\overline{\theta}_i + 2\sigma}]$.*

Proof: For the first part: whenever she gets a signal $x < \underline{\theta}_i - \sigma$, player i knows that D is dominant in all possible games $G(\theta)$ given her signal. Thus it is dominant for her to play D. Similarly, whenever $x > \overline{\theta}_i + \sigma$, it is dominant for her to play C in all possible games $G(\theta)$.

For the second part: a rationalizable strategy s is a best response of i to some strategy s_{-i} which is itself a best response of player $-i$ to an other strategy of i. The first part of the lemma implies that, $s \in [s_{\underline{\theta}_i - \sigma}, s_{\overline{\theta}_i + \sigma}]$ and $s_{-i} \in [s_{\underline{\theta}_{-i} - \sigma}, s_{\overline{\theta}_{-i} + \sigma}]$. Thus, whenever she gets a signal $x < \underline{\theta}_{-i} - 2\sigma$, player i knows that player $-i$ will play D. Because w is regular, Assumption 1 implies player i must choose to coordinate on D. Respectively, when she gets a signal $x > \underline{\theta}_{-i} + 2\sigma$, player i knows that player $-i$ will play C and coordinates on C. This proves the result. ■

Proof of Proposition 1: The proof is given for $BR^{i,L}$. Player i's best response is to cooperate if and only if $\Delta_i^w(x_i, s) > 0$. For all $x \in [\underline{\theta}_i - r/2, \overline{\theta}_i + r/2]$, we must have $\theta \in [\underline{\theta}_i - r, \overline{\theta}_i + r]$. This implies $h^i(\theta) \ge \hbar \ge 0$. Thus, when $s' \preccurlyeq s$, $\forall x \in [\underline{\theta}_i - r/2, \overline{\theta}_i + r/2]$,

$$\begin{aligned} \Delta_i^w(x,s) &= \mathbf{E}[w_{12}^i - w_{22}^i + \underbrace{(w_{11}^i - w_{21}^i - w_{12}^i + w_{22}^i)}_{=h^i(\theta)\ge 0} \mathbf{1}_{s=C}|x] \\ &\ge \mathbf{E}[w_{12}^i - w_{22}^i + (w_{11}^i - w_{21}^i - w_{12}^i + w_{22}^i)\mathbf{1}_{s'=C}|x] \ \ge\ \Delta_i^w(x,s'). \end{aligned}$$

The best response is to play C if and only if $\Delta_i^w > 0$. Thus whenever $x \in [\underline{\theta}_i - r/2, \overline{\theta}_i + r/2]$, $BR^{i,L}(s')(x) = C \Rightarrow BR^{i,L}(s)(x) = C$. Finally we know from Lemma 7 that if $x < \underline{\theta}_i - r/2 \le \underline{\theta}_i - \sigma$, then $BR^{i,L}(s')(x) = BR^{i,L}(s)(x) = D$. Similarly if, $x > \overline{\theta}_i + r/2 \ge \overline{\theta}_i + \sigma$ then $BR^{i,L}(s')(x) = BR^{i,L}(s)(x) = C$. This implies that indeed, $BR^{i,L}(s') \preccurlyeq BR^{i,L}(s)$. An identical proof holds for $BR^{i,H}$. ■

Lemma 8 (convergence of beliefs) *Define* $\Psi(x_i, u)$ *by*

$$\Psi(x_i, u) \equiv \frac{f_\varepsilon(u) f_\theta(x_i - \sigma u)}{\int_{-1}^{+1} f_\varepsilon(\tilde{u}) f_\theta(x_i - \sigma \tilde{u}) d\tilde{u}}.$$

Then, uniformly over $K \times [-1, 1]$,

(i) $\lim_{\sigma \to 0} \Psi(x_i, u) = f_\varepsilon(u)$

(ii) $\lim_{\sigma \to 0} \frac{\partial \Psi}{\partial x_i}(x_i, u) = 0.$

Proof: The proof exploits the assumption that f_θ is C^1, and that over the relevant range, $f_\theta(x_i) > 0$. The proof of (i) is straightforward. For (ii), we have that

$$\frac{\partial \Psi}{\partial x_i} = \frac{f_\varepsilon f_\theta'}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x_i - \sigma u) \mathrm{d}u} - \frac{f_\varepsilon f_\theta \int_{-1}^{1} f_\varepsilon(u) f_\theta'(x_i - \sigma u) \mathrm{d}u}{\left[\int_{-1}^{1} f_\varepsilon(u) f_\theta(x_i - \sigma u) \mathrm{d}u\right]^2}.$$

Hence,

$$\lim_{\sigma \to 0} \frac{\partial \Psi}{\partial x_i} = \frac{f_\varepsilon(u) f_\theta'(x_i)}{f_\theta(x_i)} - \frac{f_\varepsilon(u) f_\theta(x_i) f_\theta'(x_i)}{f_\theta(x_i)^2} = 0.$$

which concludes the proof. ■

Lemma 9 (bounded differences) *Consider a class of payoffs* $\Lambda_{\kappa,\nu,d,\rho,K}$ *then there exists* $M > 0$ *such that for any* $\theta \in K$, *and any payoff structure* $w \in \Lambda_{\kappa,\nu,d,\rho,K}$, *we have* $|a_w^i(\theta)| < M$ *and* $|b_w^i(\theta)| < M$.

Proof: The payoff structure w is such that a_w^i and b_w^i have zeroes in K. Since payoff functions of w have a modulus of continuity ρ, both a_w^i and b_w^i have a modulus of continuity 2ρ. Since K is compact, this implies that there exists M depending only on K and ρ such that $|a_w^i(\theta)| < M$ and $|b_w^i(\theta)| < M$. ■

Proof of Proposition 2: Consider the function $\Delta_i^w(x_i, s_x)$. A best response to s_x is characterized by the solutions of equation $\Delta_i^w(x_i, s_x) = 0$. Thus, it suffices to show that

$\Delta_i^w(\cdot, s_x)$ is strictly increasing in its first argument.

$$\Delta_i^w(x_i, s_x) = \mathbf{E}[(w_{12}^i - w_{22}^i)\mathbf{1}_{x_{-i}<x} + (w_{11}^i - w_{21}^i)\mathbf{1}_{x_{-i}>x}|x_i, s_x].$$

We have already defined $a^i(\theta) = w_{12}^i(\theta) - w_{22}^i(\theta)$ and $b^i(\theta) = w_{11}^i(\theta) - w_{21}^i(\theta)$. Denote F_ε the cumulative distribution function of ε and define $G_\varepsilon = 1 - F_\varepsilon$. We can write

$$\Delta_i^w(x_i, s_x) = \int_{-\infty}^{+\infty} \left[a^i(\theta) F_\varepsilon\left(\frac{x-\theta}{\sigma}\right) + b^i(\theta) G_\varepsilon\left(\frac{x-\theta}{\sigma}\right)\right] f(\theta|x_i)\mathrm{d}\theta. \tag{12}$$

Do the change in variable $u = \frac{x_i - \theta}{\sigma}$,

$$\begin{aligned}\Delta_i^w(x_i, s_x) &= \int_{-1}^{+1} \underbrace{\left[a^i(x_i - \sigma u) F_\varepsilon\left(\frac{x - x_i}{\sigma} + u\right) + b^i(x_i - \sigma u) G_\varepsilon\left(\frac{x - x_i}{\sigma} + u\right)\right]}_{\equiv \phi(x_i, u)} \\ &\quad \times \underbrace{\frac{f_\varepsilon(u) f_\theta(x_i - \sigma u)}{\int_{-1}^{+1} f_\varepsilon(\tilde{u}) f_\theta(x_i - \sigma \tilde{u}) \mathrm{d}\tilde{u}}}_{=\Psi(x_i, u)} \mathrm{d}u.\end{aligned}$$

This yields that,

$$\frac{\partial \Delta_i^w}{\partial x_i}(x_i, s_x) \geq \int_{-1}^{+1} \frac{\partial^- \phi}{\partial x_i} \Psi \mathrm{d}u + \int_{-1}^{+1} \phi \frac{\partial \Psi}{\partial x_i} \mathrm{d}u.$$

Observe that

$$\frac{\partial^- \phi}{\partial x_i} \geq \frac{\partial^- a_w^i}{\partial x_i} F_\varepsilon + \frac{\partial^- b^i}{\partial x_i} G_\varepsilon + \frac{1}{\sigma}(b_i - a_i) f_\varepsilon.$$

Using the assumption of strictly increasing differences in the state of the world and Lemma 1, we obtain that $\frac{\partial^- \phi}{\partial x_i} > \kappa > 0$. This, joined with Lemmas 8 and 9 yields that there exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, $w \in \Lambda_{\kappa,\nu,d,\rho,K}$ and all $x_i \in [\underline{\theta}_i(w) - \nu, \overline{\theta}_i(w) + \nu]$,

$$\frac{\partial \Delta_i^w}{\partial x_i}(x_i, s_x) \geq \kappa/2 > 0.$$

This proves the first part of the proposition. For the second part of the proposition, note that equation (12) implies that $\Delta_i^w(x_i, s_x)$ is continuous in x_i and x. The continuity and strict monotonicity of Δ_i^w imply that the solution $x_i(x)$ to $\Delta_i^w(x_i, s_x) = 0$ is continuous in x_s. ∎

Proof of Lemma 3: The proof of part (i) is straightforward. Consider a potential

equilibrium threshold x_i. If player $-i$ gets a signal $x < x_i - 2\sigma$ or $x > x_i + 2\sigma$, then she knows for sure what player i does. Because w is regular, Assumption 1 implies player $-i$ must choose the same action. This implies that if (x_i, α) is an equilibrium, then $\alpha \in [-2, 2]$. This conclude the proof of part (i).
Let us now prove part (ii). Define

$$\xi(x_i, \alpha, u) \quad = \quad a^i(x_i - \sigma u) F_\varepsilon(u + \alpha) + b^i(x_i - \sigma u) G_\varepsilon(u + \alpha) \tag{13}$$

We have that,

$$\frac{\partial \Delta_i^w}{\partial x_i} \geq \int_{-1}^{1} \left[\frac{\partial^- \xi}{\partial x_i} \Psi + \xi \frac{\partial \Psi}{\partial x_i} \right] \mathrm{d}u. \tag{14}$$

From equation (13) and by definition of $\Lambda_{\kappa,\nu,d,\rho,K}$, we obtain that

$$\frac{\partial^- \xi}{\partial x_i} \geq \frac{\partial^- a_w^i}{\partial \theta} F_\varepsilon + \frac{\partial^- b_w^i}{\partial \theta} G_\varepsilon \geq \kappa. \tag{15}$$

Moreover, we have

$$\frac{\partial \Psi}{\partial x_i} = \frac{f_\varepsilon f'_\theta}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x_i - \sigma u) \mathrm{d}u} - \frac{f_\varepsilon f_\theta \int_{-1}^{1} f_\varepsilon(u) f'_\theta(x_i - \sigma u) \mathrm{d}u}{\left[\int_{-1}^{1} f_\varepsilon(u) f_\theta(x_i - \sigma u) \mathrm{d}u \right]^2}. \tag{16}$$

Moreover, by Lemma 8 we know that uniformly over $[\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r]$,

$$\lim_{\sigma \to 0} \frac{\partial \Psi}{\partial x_i} = \frac{f_\varepsilon(u) f'_\theta(x_i)}{f_\theta(x_i)} - \frac{f_\varepsilon(u) f_\theta(x_i) f'_\theta(x_i)}{f_\theta(x_i)^2} = 0. \tag{17}$$

By Lemma 9, there exists a constant $M \in \mathbb{R}$ such that for all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$ and all $\theta \in K$,

$$|\xi_w(\theta, \alpha, u)| < |a_w^i(\theta)| + |b_w^i(\theta)| < M. \tag{18}$$

Equations (17) and (18) imply there exists $\overline{\sigma}$ small enough such that whenever $\sigma \in (0, \overline{\sigma})$, then $\left| \xi \frac{\partial \Psi}{\partial x_i} \right| \leq \kappa/2$. This and equation (15) imply that over K, $\frac{\partial \Delta_i^w}{\partial x_i} > \kappa/2$. This concludes the proof of part (ii).

We now turn to the proof of part (iii). We know from equation (5) that Δ_i^w is differentiable in α. We have,

$$\frac{\partial \Delta_i^w}{\partial \alpha} \leq \int_{-1}^{1} \frac{\partial^+ \xi}{\partial \alpha} \Psi \mathrm{d}u.$$

Moreover,

$$\begin{aligned} \frac{\partial \xi}{\partial \alpha}(x_i, \alpha, u) &= \left[a^i(x_i - \sigma u) - b^i(x_i - \sigma u)\right] f_\varepsilon(u + \alpha) \\ &= -h_w^i(x_i - \sigma u) f_\varepsilon(u + \alpha). \end{aligned}$$

By Lemma 1, for all $\theta \in [\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r]$ we have $h_w^i(\theta) \geq \hbar$. This yields,

$$\frac{\partial \xi}{\partial \alpha}(x_i, \alpha, u) \leq -\hbar f_\varepsilon(u + \alpha). \tag{19}$$

Integrating over $[-1, 1]$, we obtain

$$\frac{\partial \Delta_i^w(x_i, \alpha, \sigma)}{\partial \alpha} \leq -\hbar \int_{-1}^{1} f_\varepsilon(u + \alpha) \Psi(x_i, u) \mathrm{d}u < 0.$$

This concludes the proof of part (iii). ■

Proof of Lemma 4: All payoff functions w in $\Lambda_{\kappa,\nu,d,\rho,K}$ share a common modulus of continuity. By Lemma 9, we know that there exists M such that for all $w \in \Lambda_{\kappa,\nu,d,\rho,K}$, we have $|a_w^i| + |b_w^i| < M$ over K. Denoting by $||\cdot||_\infty$ the supremum norm, this implies that

$$|\Delta_i^w(x, \alpha, \sigma) - \Delta_i^w(x, \alpha, 0)| \leq 4\rho(\sigma) + M ||f'_\theta||_\infty \, \sigma.$$

This shows there exists $N > 0$ independent of w such that $|\Delta_i^w(x, \alpha, \sigma) - \Delta_i^w(x, \alpha, 0)| \leq N \max\{\rho(\sigma), \sigma\}$. ■

Proof of Lemma 5: A monotone equilibrium is characterized by a pair (x_i, α) such that $\Delta_i^w(x_i, \alpha, \sigma) = \Delta_{-i}^w(x_i + \alpha\sigma, -\alpha, \sigma) = 0$. We know from Lemma 7 that whenever $\sigma < \nu$ $x_i \in \cap_{i \in \{1,2\}}[\underline{\theta}_i - \nu, \overline{\theta}_i + \nu]$. Moreover, we must have $\alpha \in [-2, 2]$.

Let us first show the tighter bounds on equilibrium values of α. Define $\chi_\sigma^w(x_i, \alpha) = \Delta_i^w(x_i, \alpha, \sigma) - \Delta_{-i}^w(x_i + \alpha\sigma, -\alpha, \sigma)$. If (x_i, α) is an equilibrium, then $\chi_\sigma^w(x_i, \alpha) = 0$. At the limit where $\sigma = 0$, we have,

$$\chi_0^w(x_i, \alpha) = \int_{-1}^{1} \Big[a_w^i(x_i)F_\varepsilon(u+\alpha) + b_w^i(x_i)G_\varepsilon(u+\alpha) - a_w^{-i}(x_i)F_\varepsilon(u-\alpha) - b_w^{-i}(x_i)G_\varepsilon(u-\alpha)\Big] f_\varepsilon(u)\mathrm{d}u \tag{20}$$

Which yields,

$$\chi_0^w(x_i, -2) = b_w^i(x_i) - a_w^{-i}(x_i) \geq d\kappa \tag{21}$$

$$\chi_0^w(x_i, 2) = a_w^i(x_i) - b_w^{-i}(x_i) \leq -d\kappa \tag{22}$$

Over this range, we know there exists M dependent only on ρ and K such that $|b_w^i| + |a_w^{-i}| < M$. Moreover, f_ε is bounded over $[-1, 1]$; thus we conclude from equation (20), that there exists a constant $Q > 0$ such that $\chi_0^w(x_i, \alpha)$ is Q-Lipschitz in α. This and equations (21) and (22) imply that,

$$\chi_0^w(x_i, \alpha) \geq d\kappa - Q(\alpha + 2)$$

$$\chi_0^w(x_i, \alpha) \leq -d\kappa + Q(2 - \alpha)$$

Finally, using Lemma 4, we know there exists N depending only on ρ and K such that

$$\chi_\sigma^w(x_i, \alpha) \geq d\kappa - Q(\alpha + 2) - N\rho(\sigma) \tag{23}$$

$$\chi_\sigma^w(x_i, \alpha) \leq -d\kappa + Q(2 - \alpha) + N\rho(\sigma) \tag{24}$$

T Using equations (23) and (24) and the fact that at an equilibrium (x_i, α), we must have $\chi_\sigma^w(x_i, \alpha) = 0$, we obtain that whenever (x_i, α) is an equilibrium, then we must have

$$\alpha \in \left[-2 + \frac{d\kappa - N\rho(\sigma)}{Q}, 2 - \frac{d\kappa - N\rho(\sigma)}{Q}\right]. \tag{25}$$

Since $\rho(\cdot)$ is decreasing, this proves (i) and allows us to take for some $\overline{\sigma}$ small enough

$$\lambda \equiv \frac{d\kappa - N\rho(\overline{\sigma})}{Q} > 0.$$

From Lemma 3, we know that,

$$\frac{\partial \Delta_i^w(x_i,\alpha,\sigma)}{\partial \alpha} \leq -\hbar \int_{-1}^{1} f_\varepsilon(u+\alpha)\Psi(x_i,u)\mathrm{d}u.$$

Hence there exists $\eta > 0$ such that for all $w \in Wr$ and all $\alpha \in [-2+\lambda, 2-\lambda]$,

$$-\hbar \int_{-1}^{1} f_\varepsilon(u+\alpha) f_\varepsilon(u)\mathrm{d}u < -2\eta. \tag{26}$$

Since by Lemma 8 we know that $\Psi(x_i,u)$ converges to $f_\varepsilon(u)$ over any compact, there exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0,\overline{\sigma})$ and all $\alpha \in [-2+\lambda, 2-\lambda]$,

$$\frac{\partial \Delta_i^w(x_i,\alpha,\sigma)}{\partial \alpha} < -\eta. \tag{27}$$

This proves part (*ii*). Part (*iii*) is now straightforward:given α and x_i and σ, the function $\Delta_i^w(x_i,\alpha,\sigma)$ is 4-Lipschitz in w; this and equation (27) yields that $\alpha(x_i,w,\sigma)$ is $(\frac{4}{\eta})$-Lipschitz in w. ■

Proof of Proposition 3: Pick $\overline{\sigma}$ such that Theorem 2 holds. Note that $\Delta_i^w(x_i,\alpha,\sigma)$ is continuous in w. In fact it is 4−Lipschitz in w. This and Lemma 5 implies that the implicit function $\alpha(x_i,w)$ is strictly increasing in x_i and $(\frac{4}{\eta})$-Lipschitz in w. Thus the function $\zeta : x_i \mapsto \Delta_{-i}^w(x_i+\alpha(x_i)\sigma, -\alpha(x_i),\sigma)$ is continuous in the payoffs and strictly increasing in x_i. This implies that the solution to $\Delta_i^w(x_i+\alpha(x_i)\sigma, -\alpha(x_i),\sigma) = 0$ is a continuous function of the payoff w. This concludes the proof. ■

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